

Journey into the world of science and art in the plantigrade machine

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The site of the article: <https://community.ptc.com/t5/PTC-Mathcad-Questions/Chebyshev-Math-Journey/td-p/500239>.

"I sat down and went. In the railway carriage it is quiet, there are no pushes. "

F. M. Dostoyevsky "The Gambler"

A century and a half ago only very rich and physically healthy people could afford to travel the world. But with the advent of modern vehicles, such pleasure became available to the "broad masses of working people", and not just to the elected: got on a plane, car or train - and in a short time comfortably reached every corner of the Earth¹.

Something similar can be said about ... mathematics. Previously, only selected people could get into its jungle - people with special mathematical abilities (with a special "mathematical ear") and having the appropriate mathematical education. But at the present time the circle of such elect has significantly expanded due to the emergence of ... computer mathematical packages that facilitate the journey into the world of mathematics. Conditionally it can be said that some kind of mass mathematical tourism has appeared.

And one more important introductory remark.

Why do we study mathematics at schools and universities?

First, in order to be able to master other academic disciplines: physics, chemistry, theoretical mechanics, fluid dynamics, material resistance, engineering graphics, economics, finance, etc. That's why the course of mathematics is read at the very beginning of study at universities!

Secondly, one must always remember that mathematics is the best gymnastics (fitness) for the mind. Studying mathematics, we develop our mental abilities, which will be useful for us in solving not only purely mathematical, but also everyday problems.

And thirdly, the study of mathematics (a journey into its "jungle") is in itself an interesting and fascinating activity that can be practiced as a hobby. But without mathematical computer packages, it was almost impossible for ordinary people to do this, if, we repeat, there are no special mathematical talents and the corresponding mathematical education.

Let's illustrate these theses on simple and entertaining mechanical-mathematical examples.

¹ The words in the epigraph are pronounced by the paralyzed "granny" who "lost her legs" and which is transferred from place to place in the chair. She quietly lived in Moscow (the second half of the XIX century), but dispersed the doctors, boarded the train and "fell down like a snow on her head" to her relative - the general who was in Rutletenburg (the German city of Baden-Baden) and who was waiting for the death of "granny" and inheritance from it – see <http://ilibrary.ru/text/67/p.1/index.html>.

The Polytechnic Museum of Moscow keeps a device that has a direct relation to travel and "around the world" and "in mathematics." This is a standing Chebyshev machine (the plantigrade machine), which converts rotational motion into translational motion - see Fig. 1.

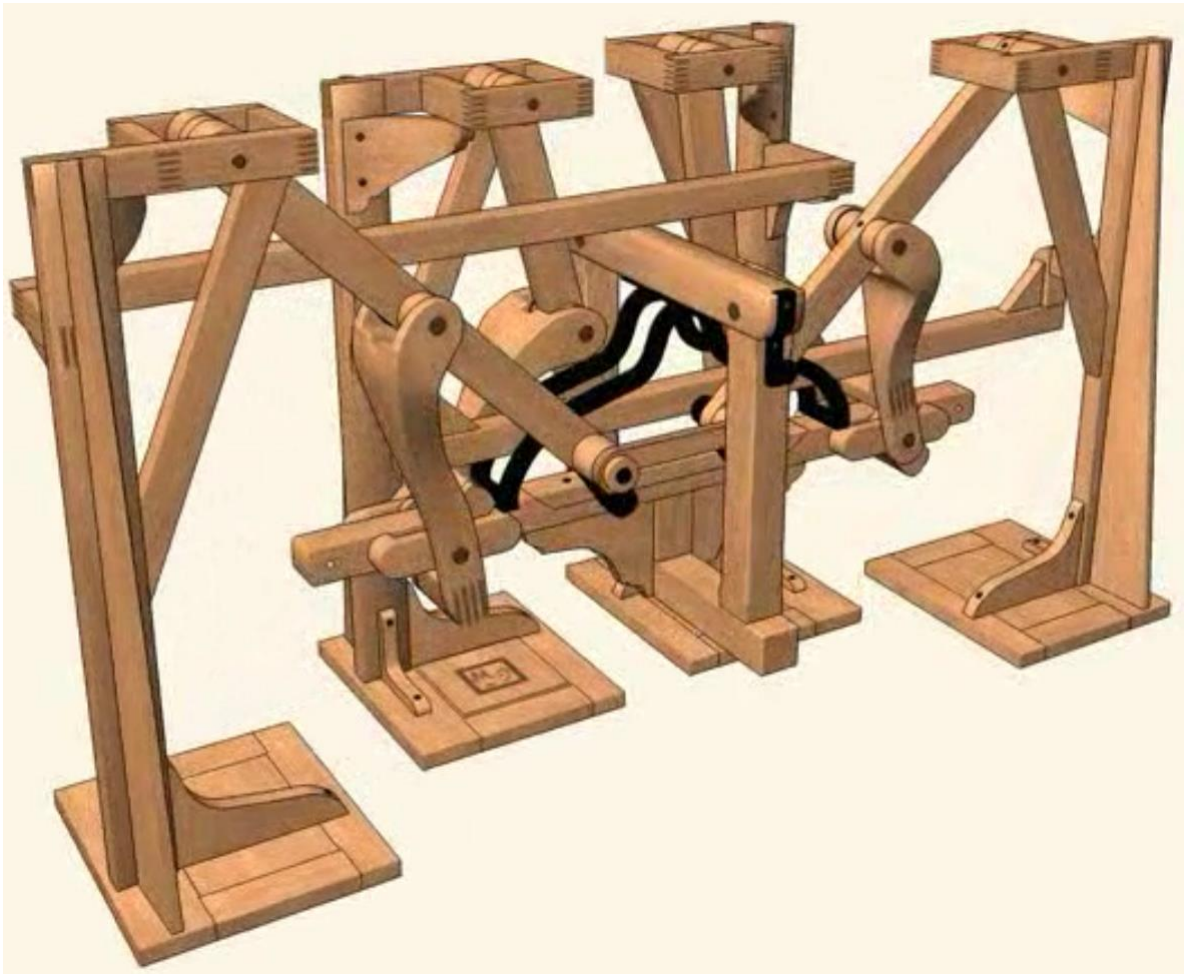


Fig. 1. Chebyshev's standing machine

Figure 1 is taken from the site of mathematical studies <http://www.etudes.ru/ru/etudes/chebyshev-plantigrade-machine>, where you can see the animation of the movement of this machine, as well as a detailed description of its device. The central link of this exotic vehicle is the node schematically shown in Fig. 2 (λ -Linkage). Points 1 and 2 are fixed, point 3 rotates around point 2, point 0 moves along an arc of a circle, and point 4 on a closed curve ("mushroom's head"), the lower part of which is close to a straight horizontal line. The movement of this point is transmitted to the "feet" of the machine shown in Fig. 1.

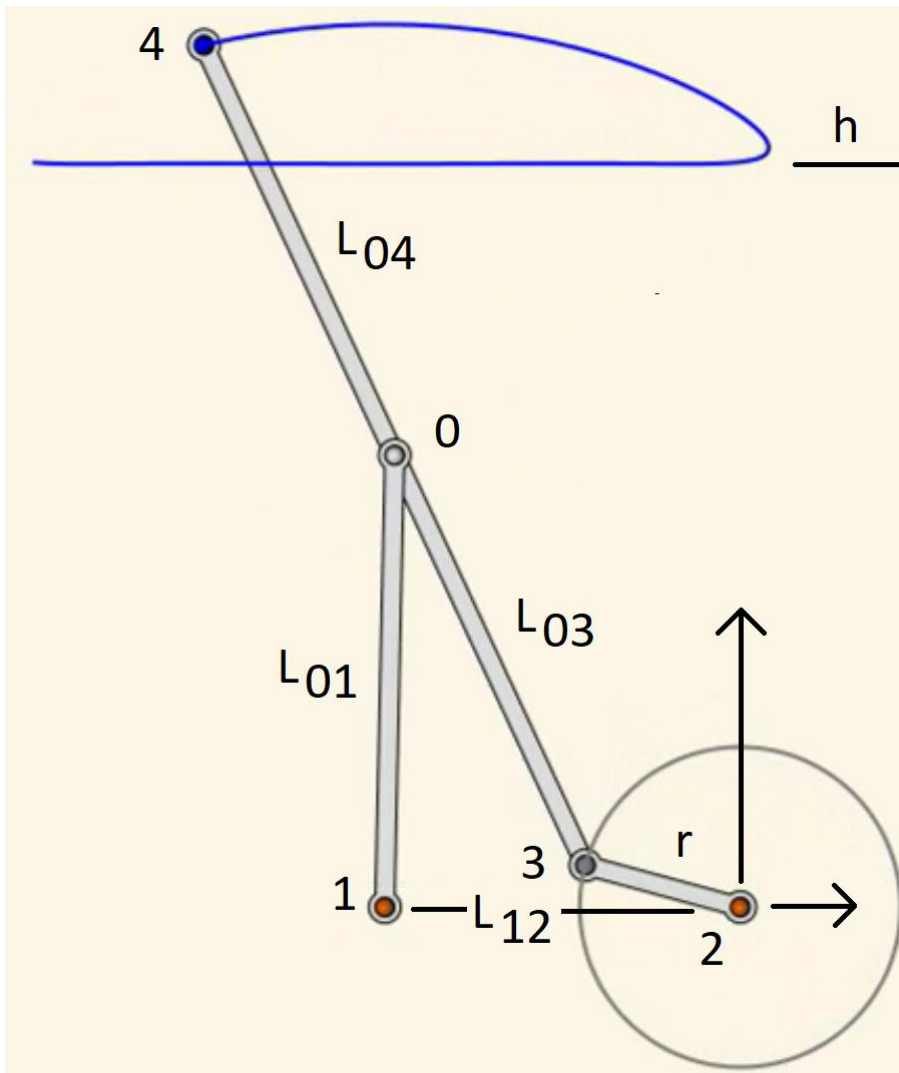


Fig. 2. Scheme of the central node of a standing machine

Such devices were made even before Chebyshev, choosing experimentally the length of the segments L_{01} , L_{03} , L_{12} и L_{04} , as well as the height h so that the standing car goes without vertical thrusts. But only our great compatriot began to apply higher mathematics for the design of such devices. We can say that Chebyshev saddled this "wooden horse" and made a trip to the World of Mathematics [1]. Let's do it [2, 3]. At us, we repeat, there can be no special mathematical talents and corresponding mathematical formation. But we have at hand a computer with a mathematical program Mathcad. We will solve the problem in a hybrid way [4] - using both analytical transformations (computer symbol mathematics) and numerical (approximate) methods of solution.

A little about numeric mathematic.

At present, numerical mathematics supplants symbolic mathematics, the mathematics of analytical evaluations in solving not only purely practical problems, but also the objectives of the curriculum. "Culprits" of this process are computers. A vivid concrete example. If we want to embody our standing machine in metal or wood (see Figure 1), then we will have to calculate not only its geometry (see Figure 2 and other figures below), but also the shape and dimensions (profiles) of its parts. They should be strong enough that the machine does not break when walking, but not too massive, so that the machine was easy on the move. Calculations of this kind are conducted within the framework of the scientific discipline

"Strength of materials". Previously, Russian students of technical universities even had such a saying: "I passed the exam on Strength of materials and can marry!". The fact is that this academic subject was very difficult to master because it was overloaded with complex empirical formulas, difficult for both understanding and for simple mechanical memorization. There was a special "terrible" formula or set of formulas for each concrete case of calculating the structural strength. Now, the finite element method (FEM) makes it possible to break a detail of any complexity into individual small (finite) elements, to the calculation of which fairly simple and understandable formulas are applied. This work became possible only after the appearance of high-speed computers with the corresponding programs. Such a revolutionary shift from analytical transformations to numerical computer calculations has affected other scientific disciplines - hydrodynamics, heat and mass transfer, etc., difficult for both understanding and simple mechanical memorization.

The essence of the calculation of a standing machine is the following: it is necessary to calculate the lengths of the links L_{01} , L_{03} , L_{12} и L_{04} , as well as the height h for a given value of the radius r so that point 4 on the lower part of its trajectory moves almost along a straight horizontal line, and a standing machine with such a central node would go as far as possible without vertical vibrations.

Figure 3 shows the beginning of the Mathcad-sheet with the calculation of the geometry of the mechanism shown in Fig. 2. In the first line of the sheet we see input of the initial data - the guess values of the lengths of the links of the central node of the standing-up machine². The value of the angle α of turning the link 2-3 is tied to the system variable FRAME, which will change from 0 to 360 and fix the frames of the animation, which we will create later. When debugging the calculation around the graph (a separate animation frame) with the image of our node (see below), we will write the FRAME statement 30, for example, (global assignment) and see the geometry of the node when turning the lever 2-3 to 30 degrees. When creating an animation, this operator will need to be disabled so that the Mathcad package can create an animation [2]. The package itself will change the value of the variable FRAME and store the animation frames in the avi container.

Points 1 and 2, we repeat, are fixed, and points 0, 3 and 4 are mobile. Their coordinates are given in the form of functions with names x_0 , y_0 , x_3 , y_3 , x_4 and y_4 with arguments α , r , L_{12} , L_{03} , L_{01} и L_{04} . Working with functions, rather than with the same scalar values, allows us, firstly, to create graphs and animations with the drawing of the trajectories of all points³ and, secondly, to carry out optimization - to find the optimal lengths of links L_{12} , L_{03} , L_{01} и L_{04} , as well as the heights h , which ensure the maximum rectilinear motion of the standing machine - movement without "jerks".

² They are entered with a unit of length meter. But this is a convention. There may be other meanings, and other units of length (centimeters, fouts, inches or arshins (аршин) with vershoks (вершок), which were in use in Russia at the time of Chebyshev: 3 arshins = 7 fouts, 16 vershok = 1 arshin [6]). The main thing is that this method allows you to control the dimensions, enter and output lengths with the necessary units of length.

³ You can find many drawings and animations of various kinematic devices on the Internet. But all of them, as a rule, are given without drawing trajectories of the movement of nodal points.

$$r := 15m \quad L_{12} := 35m \quad L_{03} := 35m \quad L_{01} := 35m \quad L_{04} := 35m \quad h := 55m \quad \alpha := \text{FRAME deg}$$

$$x_1(L_{12}) := -L_{12} \quad y_1 := 0m \quad x_2 := 0m \quad y_2 := 0m \quad x_3(\alpha, r) := r \cdot \cos(\alpha) \quad y_3(\alpha, r) := r \cdot \sin(\alpha)$$

$$L_{03}^2 = (x_0(\alpha, r, L_{12}, L_{03}, L_{01}) - x_3(\alpha, r))^2 + (y_0(\alpha, r, L_{12}, L_{03}, L_{01}) - y_3(\alpha, r))^2$$

$$L_{01}^2 = (x_0(\alpha, r, L_{12}, L_{03}, L_{01}) - x_1(L_{12}))^2 + y_0(\alpha, r, L_{12}, L_{03}, L_{01})^2$$

$$x_0(\alpha, r, L_{12}, L_{03}, L_{01}) := \frac{x_3(\alpha, r)^2 + L_{01}^2 - L_{03}^2 + y_3(\alpha, r)^2 - x_1(L_{12})^2 - \frac{y_3(\alpha, r) \cdot [y_3(\alpha, r)^3 - x_1(L_{12}) \cdot \sqrt{L_{01}^2 + 2 \cdot L_{01} \cdot L_{03} + L_{03}^2 - x_1(L_{12})^2 + 2 \cdot x_1(L_{12}) \cdot x_3(\alpha, r)}]}{2 \cdot y_3(\alpha, r)}}{2 \cdot y_3(\alpha, r)}$$

$$y_0(\alpha, r, L_{12}, L_{03}, L_{01}) := \frac{y_3(\alpha, r)^3 - x_1(L_{12}) \cdot \sqrt{L_{01}^2 + 2 \cdot L_{01} \cdot L_{03} + L_{03}^2 - x_1(L_{12})^2 + 2 \cdot x_1(L_{12}) \cdot x_3(\alpha, r)}}{2 \cdot y_3(\alpha, r)}$$

$$a(\alpha, r, L_{12}, L_{03}, L_{01}) := \frac{x_3(\alpha, r) \cdot y_0(\alpha, r, L_{12}, L_{03}, L_{01}) - y_3(\alpha, r) \cdot x_0(\alpha, r, L_{12}, L_{03}, L_{01})}{x_3(\alpha, r) - x_0(\alpha, r, L_{12}, L_{03}, L_{01})}$$

$$b(\alpha, r, L_{12}, L_{03}, L_{01}) := \frac{y_3(\alpha, r) - y_0(\alpha, r, L_{12}, L_{03}, L_{01})}{x_3(\alpha, r) - x_0(\alpha, r, L_{12}, L_{03}, L_{01})}$$

$$x_4(\alpha, r, L_{12}, L_{03}, L_{01}, L_{04}) := x_0(\alpha, r, L_{12}, L_{03}, L_{01}) + \frac{L_{04}}{L_{03}} \cdot (x_0(\alpha, r, L_{12}, L_{03}, L_{01}) - x_3(\alpha, r))$$

$$y_4(\alpha, r, L_{12}, L_{03}, L_{01}, L_{04}) := a(\alpha, r, L_{12}, L_{03}, L_{01}) + b(\alpha, r, L_{12}, L_{03}, L_{01}) \cdot x_4(\alpha, r, L_{12}, L_{03}, L_{01}, L_{04})$$

Fig. 3. Beginning of calculation of a standing machine

The position of point 0 (the "central" point of our mechanism) is determined through the analytical (symbolic) solution of the system of two algebraic nonlinear equations written on lines 2 and 3 of Figure 3. These equations were solved symbolically in a separate Mathcad-sheet (it is also stored on the article's website) . The system of equations has two roots, one of which was transferred to the sheet in Fig. 3 and formed functions with names x_0 and y_0 . The second extra root captures the geometry of the linkage shown in Fig. 2, when point 0 is not higher, but lower than points 1 and 3. These roots (algebraic expressions) are very voluminous: in Fig. 3 shows only their begins⁴.

The point 4 is on a straight line passing through points 0 and 3. To determine the coordinates of the point 4, use the equation of the straight line $y = a + b \cdot x$, whose coefficients a and b are also symbolically determined in a separate document through the solution of a system of two algebraic equations and carried over in the document in Fig. 3. This Mathcad-sheet is also posted on the article's website.

After the coordinates of all five points of our kinematic scheme have been specified or calculated, it is easy to display it on the graph - see Fig. 4.

⁴ In the calculation, it could be simpler to use not a symbolic mathematics and not user functions, but numerical mathematics (the Given-Find block) and scalar user values. But in this case, we repeat, it would be impossible to draw the trajectories of the points movement, create an animation with the display of the trajectories of the points movement and optimization.

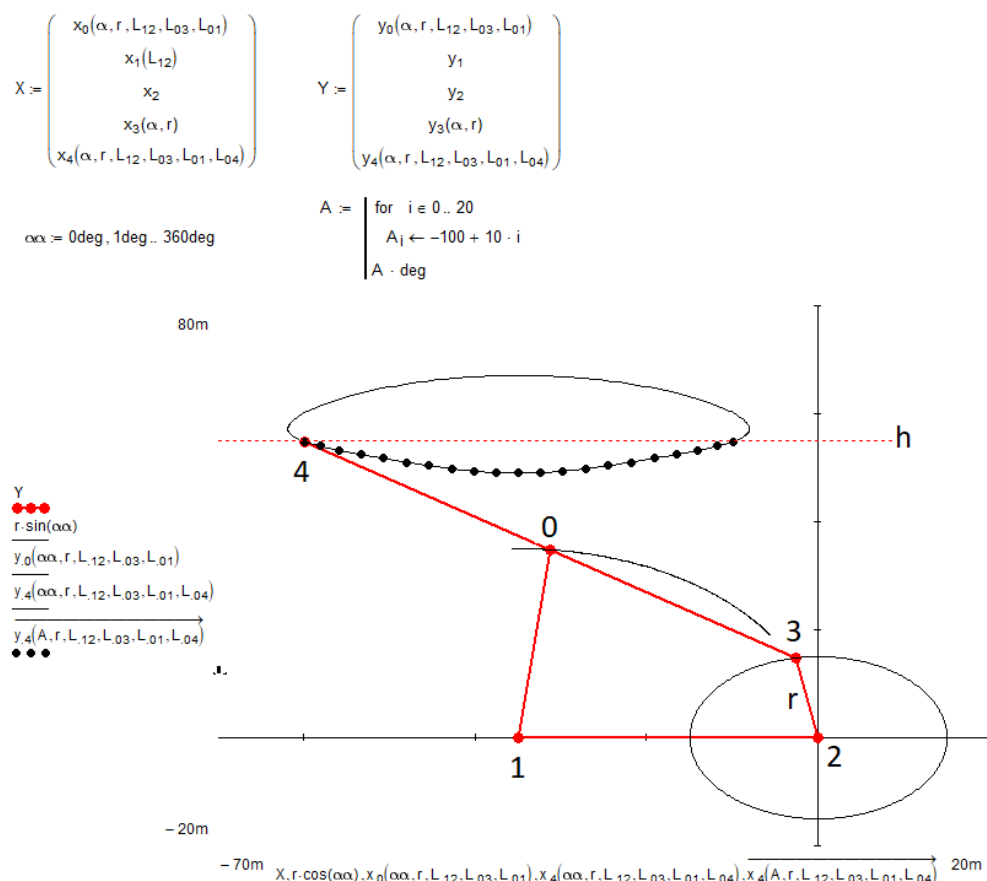


Fig. 4. The non-optimized central unit of the Chebyshev's standing machine

We can see on the plot on the Fig. 4⁵, that the given values of the variables L_{12} , L_{03} , L_{01} и L_{04} and h (see the beginning of Figure 3) are far from optimal and our step-standing car during the movement will strongly rock the rider. Как этого избежать? Pafnuty Chebyshev (https://en.wikipedia.org/wiki/Pafnuty_Chebyshev) took paper and a pen, strained his entire mathematical talent, made a bunch of analytical evaluations and eventually found the answer: a set of complex formulas by which you can calculate the optimal configuration of the central node of a standing machine. This work also resulted in the article [1]. We do not have such a mathematical talent, but there are optimization tools built into Mathcad⁶. А в настоящее время задачи такого рода, повторяем, все чаще и чаще решаются не аналитически, а численно с использованием компьютера.

On the lower part of the "mushroom cap" (Fig. 4), points are marked that mark the rotation of the link 2-3 to -100, -90, -80, ... 0, 10, 20, ..., 90 and 100 angular degrees. These angles are stored in a program-created vector with the name A . You have to make sure that these points are not in an arc, but in a straight line. For this it is easy, for example, to minimize the sum of the squares of the deviations of these points from the horizontal line!

⁵ It is slightly stretched horizontally. This happens when the graph is edited - click on it with the mouse.

⁶ By the way, the Tcheb function is built in Mathcad and other popular mathematical packages. The Tcheb function returns the Chebyshev polynomial. Because of this and, of course, for other reasons, Pafnuty Chebyshev (Chebyshev) became abroad the most famous Russian mathematician.

$$S(L_{12}, L_{03}, L_{01}, L_{04}, h) := \sum_{i=0}^{20} (y_4(A_i, r, L_{12}, L_{03}, L_{01}, L_{04}) - h)^2$$

$$S(L_{12}, L_{03}, L_{01}, L_{04}, h) = 341.443 \text{ m}^2$$

$$\begin{pmatrix} L_{12} \\ L_{03} \\ L_{01} \\ L_{04} \\ h \end{pmatrix} := \text{Minimize}(S, L_{12}, L_{03}, L_{01}, L_{04}, h) = \begin{pmatrix} 31.233 \\ 39.39 \\ 39.384 \\ 39.362 \\ 63.804 \end{pmatrix} \text{ m}$$

$$S(L_{12}, L_{03}, L_{01}, L_{04}, h) = 0.035 \text{ m}^2$$

Рис. 5. Оптимизация стопоходящей машины

Figure 5 shows how the objective function with the name S and with five arguments - optimization parameters is specified. The function S returns this same sum of squares of deviations of points from a straight line. With the optimization parameters specified by the first line in Fig. 3, this amount is equal to 341.5 m². You can ask the Mathcad package to change the optimization parameters L₁₂, L₀₃, L₀₁ and L₀₄ and h so that this amount becomes minimal. This "request" is performed by the Minimize function built into Mathcad: the sum of the squares of deviations of points from the line became 0.035 m², and our standing machine began to go almost without pitching - see Fig. 6.

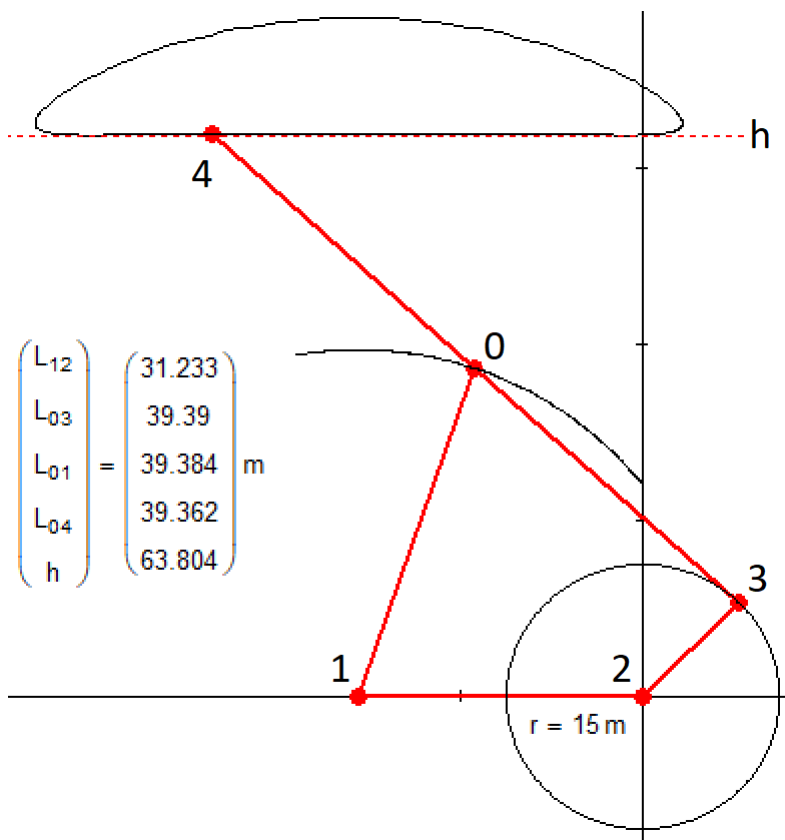


Fig. 6. Optimized central linkage of a standing Chebyshev's machine

Figure 6 shows that the values of variables L_{03} , L_{01} and L_{04} after optimization became almost equal to each other. The conclusion is that they are in fact equal, and that Chebyshev has proved this analytically long ago. So the number of independent parameters of our optimization problem can be reduced from five to three. This reduction is important for manual analytical calculations. Today, when we increasingly solve similar problems on the computer, this optimizing optimization goes to the background.

The Figure 7 shows four frames of the animation for different values of the angle of rotation α . The same animation, we repeat, you can see on the site articles <https://community.ptc.com/t5/PTC-Mathcad-Questions/Chebyshev-Math-Journey/td-p/500239>. There you can find the corresponding Mathcad-sheets for its two versions - Mathcad 15 and Mathcad Prime 4.

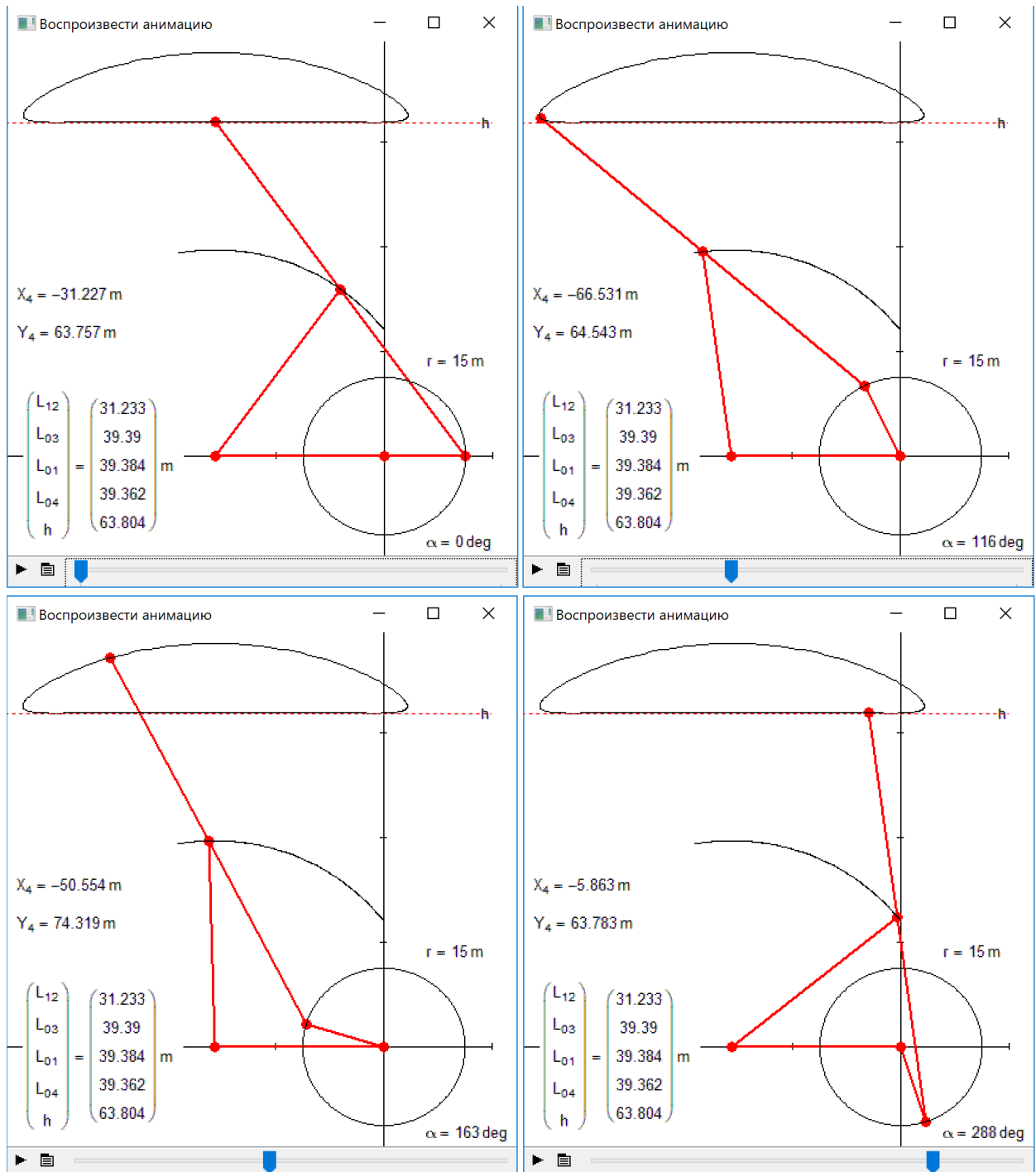


Fig. 7. Frames of animation of the central node of the standing Chebyshev's machine

Go ahead.

On a standing car you can travel not only to the world of science (mathematics and mechanics - see above), but also to the world of... art, without having the appropriate talents. And with the advent of computers, in particular computer graphics, new forms of visual art emerged, accessible to the same "broad masses of working people". We will propose one more form of fine art connected with our kinematic devices!

The Figure 8 shows how the two links (arms) pivotally mounted at one end at points 1 and 2, and revolving around these points, moving the lever the other three (one of which slides in a bearing 3)... and draw a face funny. It is possible to change the parameters of this kinematic linkage and ... make the face look more cute or more amusing...

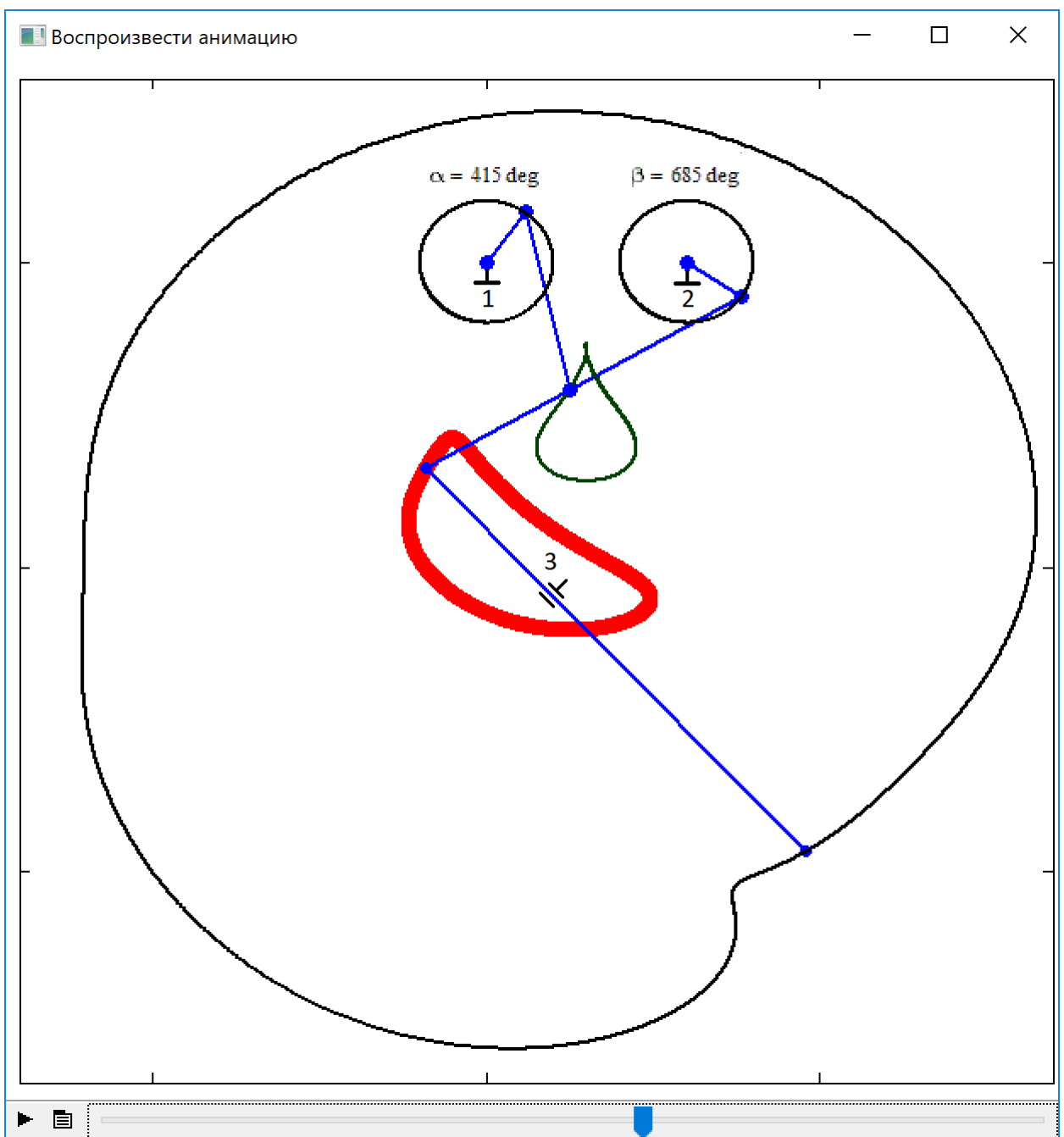


Fig. 8. Drawing a funny face with a five-link kinematic device

If one "eye" in Figure 8 is replaced the link rotating around a fixed point to a more complex device - to a cross-shaped knot, then you can draw ... a bird (The Bluebird of Happiness) - see Fig. 9.

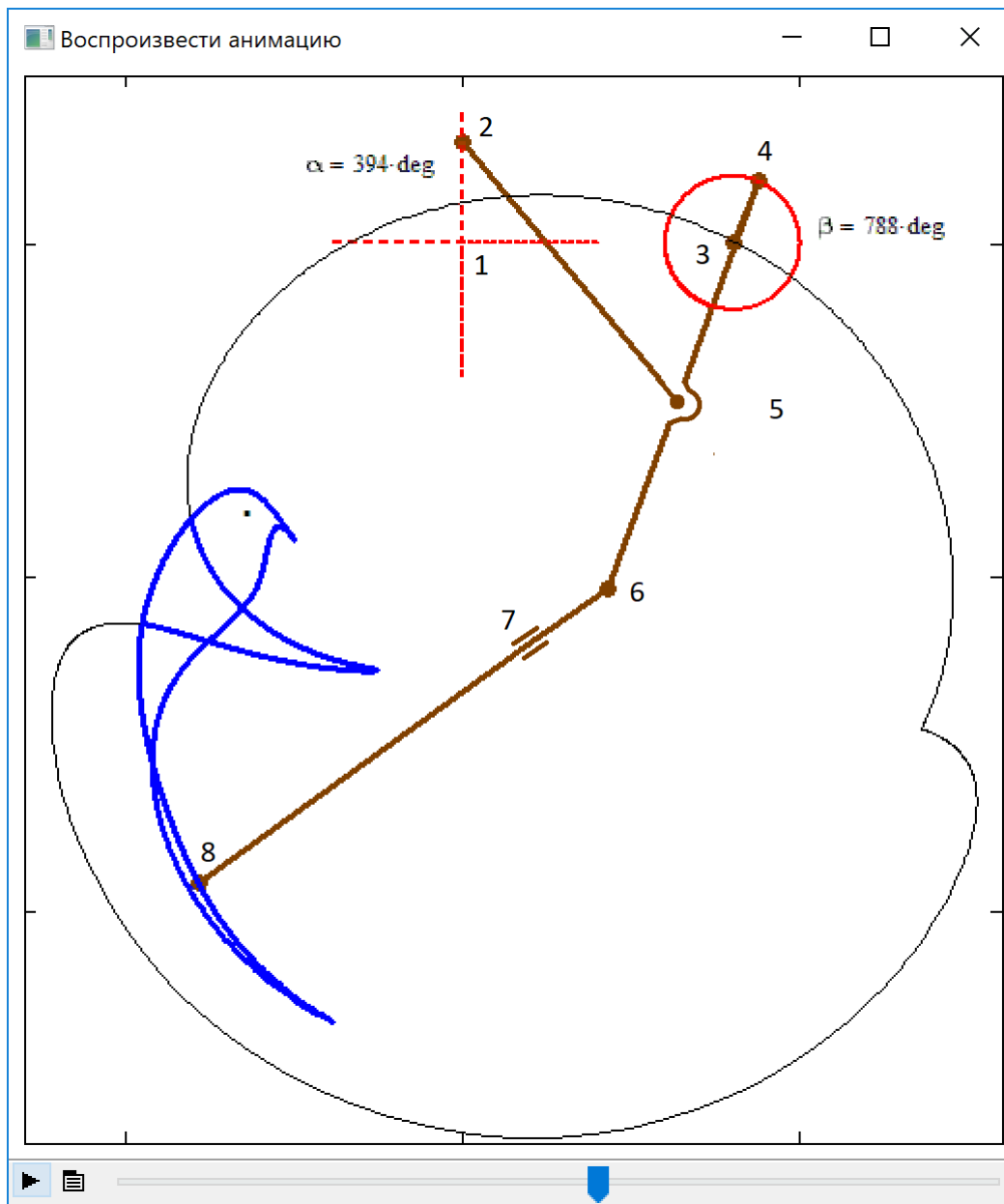


Fig. 9. Drawing a bird with the help of a kinematic linkage

Points 1, 3 and 7 are fixed. Point 2 moves right-left, up-down, left-right and up-down, drawing a cross. Point 4 moves in a circular orbit. Points 5 and 6 write out intricate trajectories that can be seen on the article's website. The final point 8 draws our blue bird.

Figure 10 shows a more complex linkage mechanism that depicts a certain lace [5]: to the four links that rotate around the four points, other articulated links are attached, the nodes of which draw closed curves. To the design, you can add new links and weave "lace" - see the site <https://community.ptc.com/t5/PTC-Mathcad-Questions/New-art-form-with-Mathcad/mp/537923> with the corresponding drawings, animations and Mathcad-files..

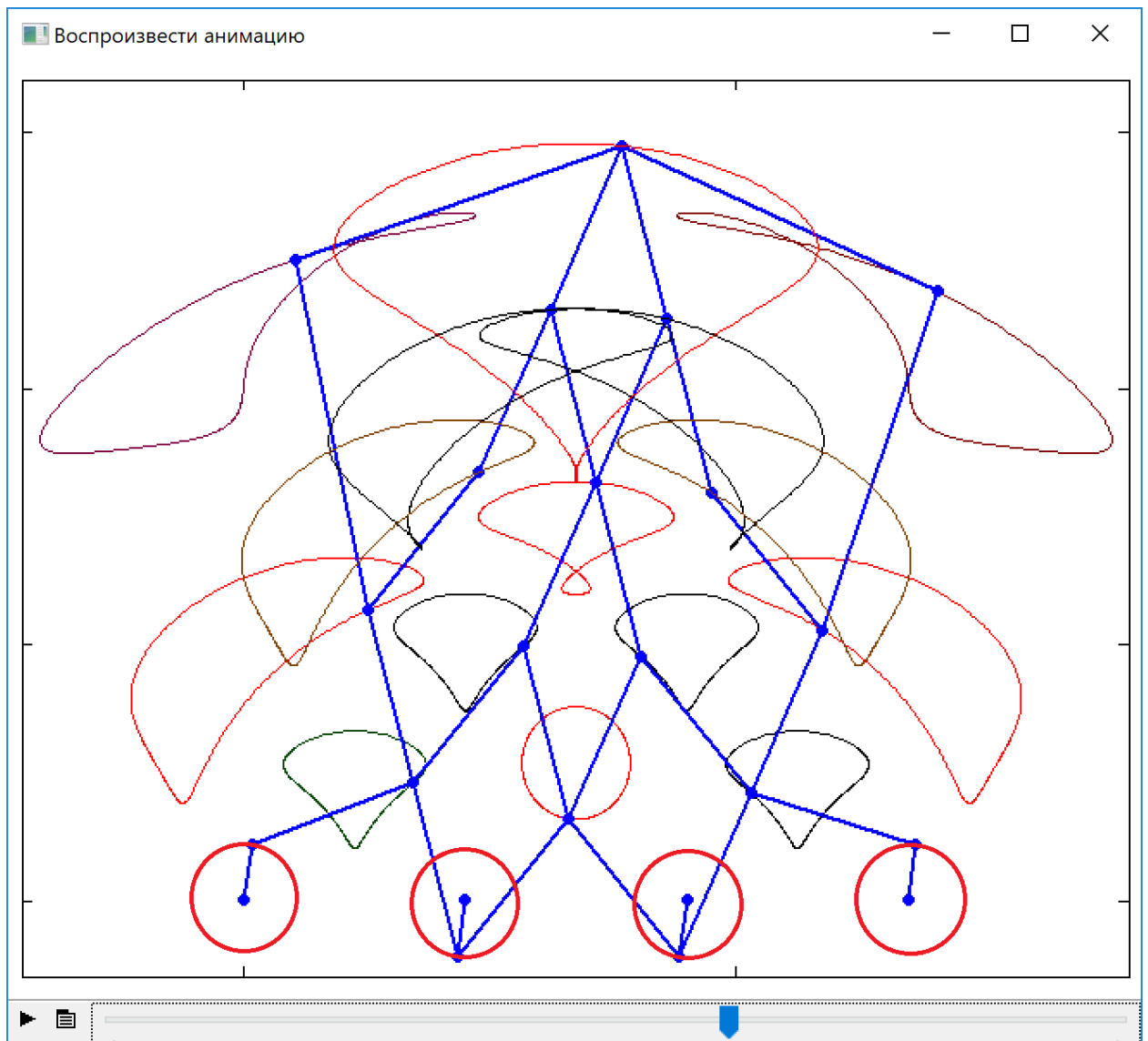


Fig. 10. Lace of trajectories of knots of the kinematic device

The number of levers that rotate around points can be increased, the radii of rotation, the speed of rotation, the lengths of the connected levers, and so on can be changed. But you can limit yourself to only two initial дшты, but rotate them not only in a circle, but also in ... "square" - see Fig. 10.

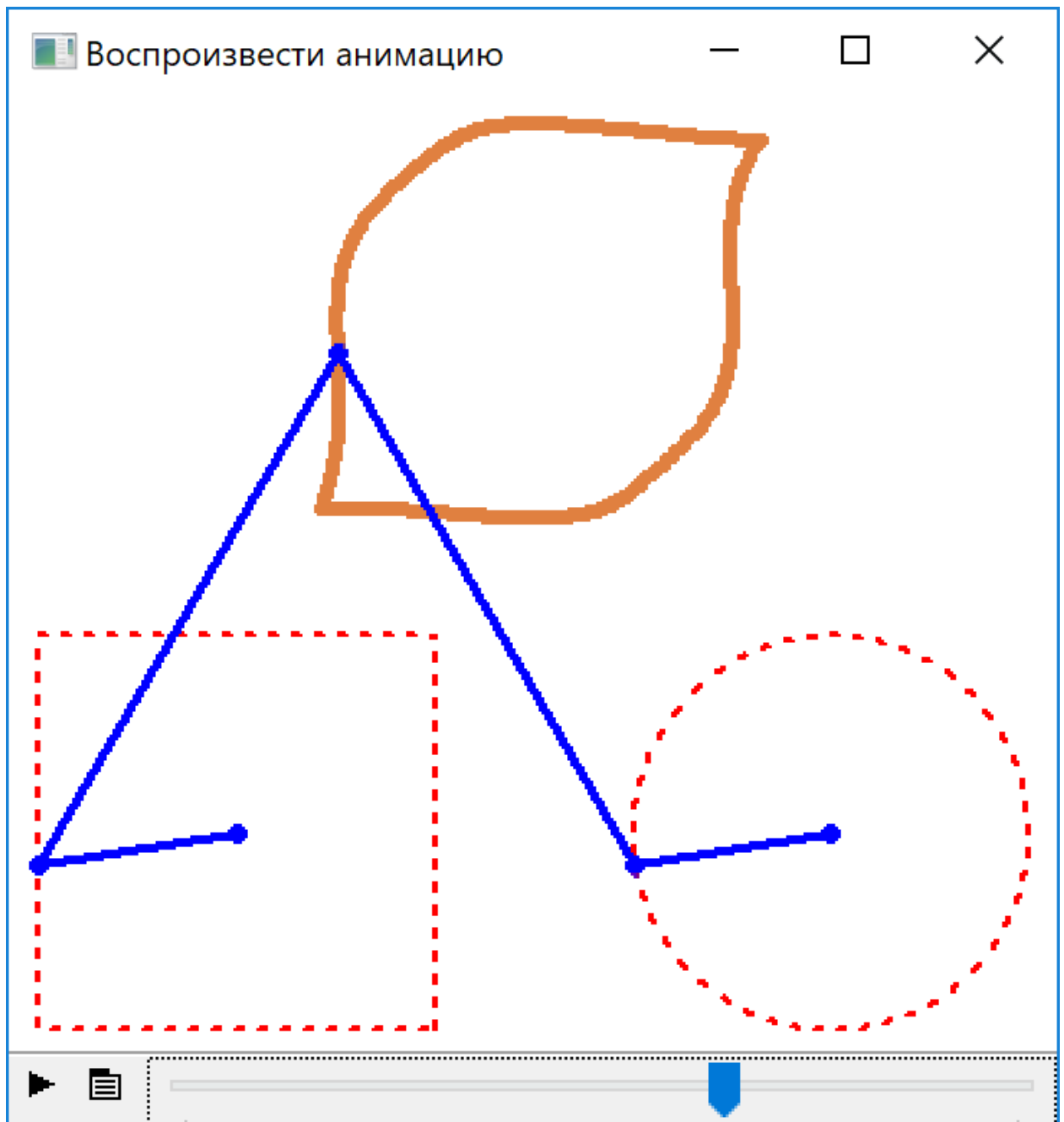


Fig. 11. Drawing a contour of a lemon with the help of a kinematic device

The links shown in figure 11 draw a contour of... one lemon. Instead of a square, you can use a triangle or other polygons and build entertaining and in some way beautiful closed curves. These attempts by the author and his colleagues are shown on the forum with the name "Circle + square = lemon" (<https://community.ptc.com/t5/PTC-Mathcad-Questions/Square-circle-lemon/m-p/538487>).

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