

Teaching Mathematics with Mathematical Software

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Synopsis

The history of contemporary mathematical education is the history of a struggle against computers and IT. As a result specially selected simplified math problems are used while teaching. Just as it was a hundred years ago, contemporary students are forced to memorize a lot of rules and theorems in order to solve math problems. But we know that today they can get the same results using simple computer calculations. Information technologies can (and in this paper we argue that they should) change the traditional methods of solving mathematical problems. Here we share some simple problems that helped engineering students learn the basics of mathematics and computer science and even enjoy the learning process. In particular we point out that the ability to visualize solutions is very important in most contexts, and modern mathematical software packages offer users convenient and simple tools of visualization and even animation. Including them on our pedagogical team, we can significantly increase our students' understanding of the basic concepts and theorems of mathematics.

1. Digital technology

The use of digital technologies to assess the knowledge has long been recognized as effective, despite the fact that it has its own peculiarities. But to convince teachers to use mathematical packages in the practical classes in mathematics is not an easy task [16, 17]. But we will try to do it.

The history of contemporary mathematical education in schools and universities, among other things, is the history of a struggle against computers and IT. Not so long ago the use of calculators in math classes was forbidden. Later mathematical software for the solution of more complex tasks (Mathematica, Maple, Matlab, Mathcad, SMath, Derive, etc.) was not recommended for use by students, (and of course was extremely expensive for researchers).

Instructors who forbade the use of such tools were sure that understanding the process of solving a given problem is the most important aspect of learning mathematics. They thought that while studying, the final result is less useful and less interesting than the method by which it is obtained. It seemed, to them, that the absence of any individual mental effort on the computational side of the problem turned the student into a computer appendage, capable only of clicking to obtain the answer produced by a machine. These kinds of opinions led to the use in practice sessions and seminars of specially selected, simplified (and oftentimes contrived) examples which had been invented long before the emergence of computers.

But today's high school students and college students do not understand this point of view and consider it outdated and unsustainable. Moreover, they cannot imagine school without a computer.

There are somewhat approachable arguments offered by supporters of the traditional pedagogy of mathematics. Some say for instance that oral exercises and manual calculations provide good gymnastics for the mind. To that, we say: well, in that case, the calculator can serve as a supplementary sport equipment. At least occasionally, it can be used as an acceptable tool for intermediate cumbersome calculations.

The most widespread argument is that we should be able to do calculations in our minds. If that is too difficult we are permitted do it with a pencil and a paper. If that is not enough then we can use the calculator, and so on. But, by analogy, if we cannot produce fire with friction then we can use matches or lighters; if in the forest we can't determine the direction of north by looking at trees then we can use a compass or a navigator, and so on. Unfortunately (or fortunately!) with the advance of civilization we have forgotten how to make fire by friction and how to find the direction of north by the moss on the trees. So we use the technical tools.

Some teachers of arithmetic will never tire of repeating to children that if they are able to count and compute quickly and accurately without a calculator, then they will never be shortchanged. But this argument is no longer valid today. Nowadays most of us (and our students!) have calculators integrated into our smart phones or compact tablet computers.

Opponents of the use of any type of computers for learning mathematics in schools and universities have their own, often hidden agenda.

Firstly, many school teachers and university professors, unfortunately, simply do not know how to work with mathematical software. They have mastered the computer at the office software level (word processing, spreadsheet, e-mail, browsing the Internet). But they do not want to go further, justifying it by the fact that, well, these programs are harmful to the learning process (see above).

Secondly, the implementation of these programs in the educational process requires a radical revision of the content and methods of teaching. Mathematics textbooks and collections of problems ought to be rewritten or at least substantially reprocessed [18].

Thirdly, the above-mentioned computer programs are quite expensive. They are not always affordable for schools and colleges. Additionally, such programs need to be on personal computers (tablet PCs, smartphones) for pupils and students to do their homework. However, most companies that develop math programs give substantial discounts to educational institutions and in some cases they share the program for free. (It is evident that, pupils and students, who master the free program, will buy it for themselves, or will ask their employers to buy the same program.)

Teachers should not complain about the high cost of programs or incorrect work (if they deal with pirated copies [11]). They should look for ways to solve the problem by direct contact with the software developers and their dealers. In addition, there are free versions of some programs. For example, the firm PTC - developer of the program Mathcad (www.ptc.com) - gives the opportunity to work first with the full version of Mathcad Prime, and then after a certain period, with its shorter version of Mathcad Express, which still allows one to solve quite complex mathematical problems. Also, the program SMATH ("Russian" Mathcad) is available for free download from www.smath.info.

The view expressed above was also presented in [13], which was devoted to the teaching of physics. In this article we are concerned with mathematics.

2. Geometry

Several events prompted the authors to write this article. The first event was when one of us helped his granddaughter solve the following problem in mathematics: One side of the triangle has length 12 cm. The angle between this side and the neighbor side of the triangle equals to 120° . The side that is opposite to this angle is equal to 28 cm. Find the length of the third side c of the triangle and the distance h between the apex of the given angle and the second side. Immediately he sat down at the computer, wrote the system of six algebraic equations, and solved it without problems in Mathcad using a solve block (see Figure 1).

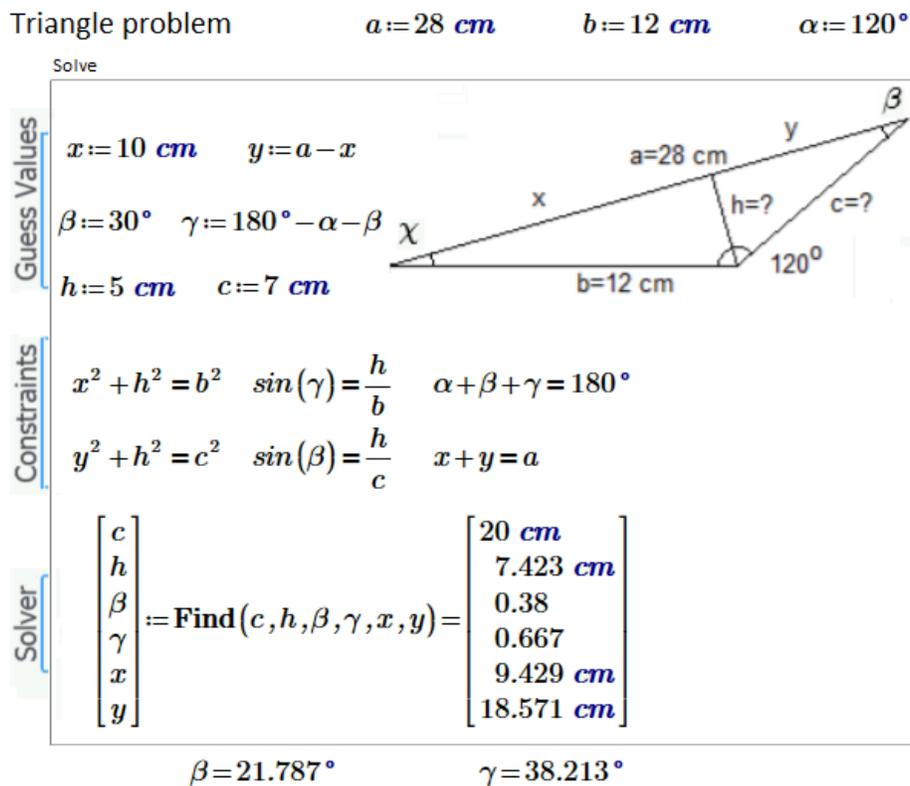


Figure 1: The triangle problem - solution using a system of algebraic equations (Mathcad).

When we put the initial approximations to the solution into the Mathcad solve bloc, the program writes the simultaneous equations and calls the function Find, which returns the values of the unknowns. In the solution shown in Figure 1, the KISS-principle¹ approach to programming is displayed.

Of course, the problem could be reduced to five equations. For example, we can replace y by $-x$. Then we can designate $180^\circ - \alpha - \beta$ by z and reduce the number of equations to four. You can even replace the two equations involving sines by the single equation, $b \cdot \sin(180^\circ - \alpha - \beta) = c \cdot \sin(\beta)$, etcetera. But if we do so we shall obscure the matter. Replacement, leading to a decrease in the number of equations, impairs understanding. And for the computer, as opposed to a person, there's not much difference between five or six equations. If the computer struggles to solve a big number of equations, then some simplification can help. But now we do not need that.

So, we have found a solution. You can verify this by substituting the solution into the equation. You obtain an identity.

When the author showed this result to his granddaughter, she said that such problems were not solved this way at school. Students had to apply the cosine theorem, which they were learning for almost three months. The grandfather did remember that there was such a theorem, but he had completely forgotten what it was. His granddaughter reminded him (in addition he looked it up on http://en.wikipedia.org/wiki/Law_of_cosines) and he rewrote the solution of the problem—see Figure 2 on the next page.

It was clear that granddaughter and her classmates are learning to solve quadratic equations (that is the essence of the problem of our triangle), but they are not allowed to use the computer or the Internet in order to find solutions for systems of equations—linear and non-linear, algebraic and differential (e.g. the differential equations discussed in [13]).

But many school and university problems in mathematics, physics, chemistry and other disciplines need solving systems of equations. It seems that the student must understand the essence of a problem in math, physics or chemistry. Then he must be able to construct a system of equations. Then he can solve and check it by computer. But no! Pupils and students are forced to memorize a bunch of rules and theorems, which are the standard solutions of these equations and systems. Using the substitutions in our triangle

¹KISS is an abbreviation of the English “Keep It Simple, Stupid”.

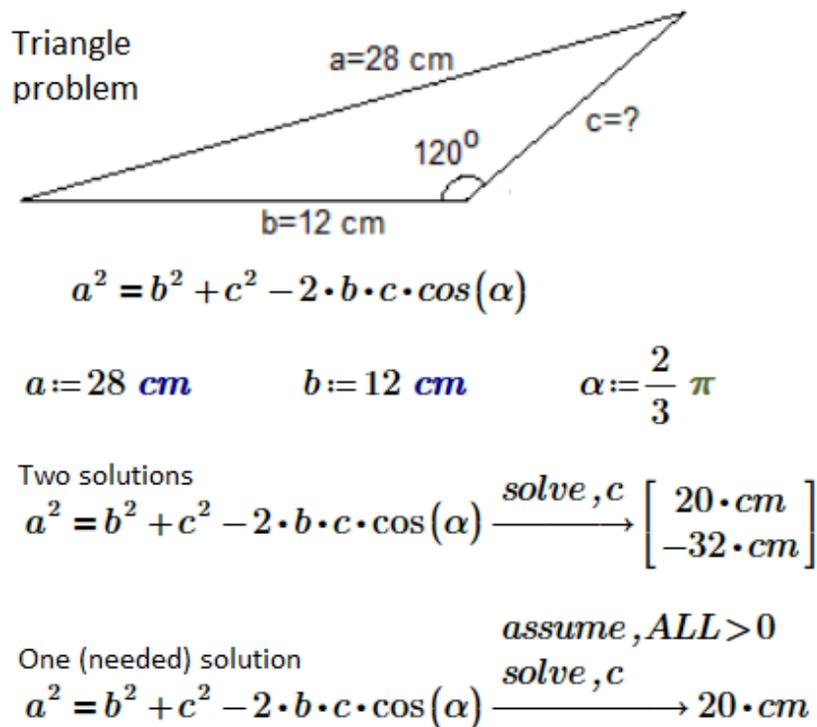


Figure 2: The triangle problem: the solution of the quadratic equation (Mathcad).

problem with six equations (Figure 1) we can get a single quadratic equation describing the cosine theorem. But you do not need to do it yourself. You can entrust the process to the computer.

The problem in Figure 1 is completely solved. The length of the third side of the triangle and one of its heights were found. But the solution in Figure 2 is incomplete. You need to find the height of the triangle. In Figure 1 the Pythagorean Theorem is used directly. Everyone knows the Pythagorean Theorem, but not all of us know the cosine theorem. The solution process displayed in Figure 1 is much more universal than the one in Figure 2. Any triangle or even a polygon one can divide into separate right-angled triangles, represent them by systems of equations and solve by computer.

We know a lot of students who are categorically prohibited to use the computer to perform calculations, or for constructing a model of a particular course project. Many teachers still say that students must do their work only with pen and paper. But the computer allows us to abandon many of the

“good old” problems and come up with some that are new, more complex, more interesting and closer to real life.

The problem of the triangle goes back to ancient Greece to the heyday of Euclidean geometry. At that time every year people had to measure and share their land.

Earlier, education in Russia was divided into classical and real. The classical school of old Russia was focused on study of Latin and Greek languages. The middle real schools were aimed at learning natural science problems. But the echo of “classicism” in education is evident in the modern school approach to the teaching of mathematics. Thus, the problem of the triangle does not use modern methods of solution. We teach only those which the ancient Greeks used. Is it good or bad? That is the question that we raise in this article.

Modern computer methods must not be ignored at school. The fact that schoolchildren and students ought to memorize a set of rules and theorems must not prevent them from tackling more complex nonstandard tasks, where the emphasis should be done on “mathematics, physics, chemistry, etc”.

The second event that prompted this article occurred as follows. A few years ago, one of us had the opportunity to deliver lectures and to conduct workshops on science at the evening department. Students were grownups and quite mature. They worked in Moscow energy companies like engineers but they had not the higher education. Fortunately, these students were very good. They came to study in Moscow Power Engineering Institute, not only to get the diploma, allowing them greater career progression. These students needed the knowledge and greater understanding of the complex processes of production, transmission and consumption of heat and electricity with which they had to deal at work.

At first the course on Computer Science was taught in the usual way. But this was not exactly what students needed and what they could learn. They were not afraid of computers because they had long since mastered them at their work. But they were very frightened of mathematical analysis: limits, derivatives, integrals, and so on.

The math teacher and the first author decided to combine efforts and help the students not just to learn the basics of Mathematics and Computer Science, but also to get pleasure from it. We usually keep telling our students that they should get not only the knowledge and skills. They also should

get joy and satisfaction from study, as well as from any other difficult but fruitful work. They should have fun. We want to tell all students: *Even if you have dreamed to become an artist or a pilot, but instead you attend the Moscow Power Engineering Institute, do not worry! Make an effort - love a school and your future specialty: enjoy! Without pleasure, even the most prestigious and highly paid work can poison life.*

3. Optimization

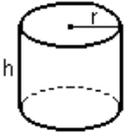
Let us return to mathematics and computers.

The first author decided to remake the syllabus of computer science, or rather, change the list of examples that illustrate the development of modern computer-based data processing. The students had a trouble with mathematical analysis, because they hardly knew the basic theory of limits, differentials, integrals, etc. So, it was difficult for students to apply their knowledge of math even to solving the simplest problems. Also they had no idea about use of these powerful math tools in their engineering work. The implication was that applications of math to engineering problems would be given later in special courses. The author decided to remedy this situation. He proposed the solving some simple engineering problems connected with the basic concepts of mathematical analysis using computer program.

These students often see large containers (tanks) of the circular cylindrical shape for the storage of fuel oil (fuel for power plants) or water (working fluid of the steam turbine generating units and the coolant for heat networks). They stand at the territory of the Moscow Combined Heat and Power (CHP) stations. Such tanks with gasoline or diesel fuel can be found at major gas stations. Few people think about the proportions of such tanks, where one and the same volume of liquid can be stored in either the high and narrow or low and wide cylinders. Using mathematics and computer software let's show, that $2r = h$ when the surface area of such a container of a fixed volume is minimized (where r is the cylinder radius and h is its height). We, like students, can solve this problem using the program Mathcad [6, 12, 14] and the tools of mathematics which students usually learn from their lessons on differential calculus (see Figure 3 on the next page.)

The well-known formulas for the calculation of the volume V and the total surface area S of the cylinder with base radius r and height h are well known. Using the tools of symbolic computation in Mathcad, it is easy to

Cylinder with bottom and top



Surface S : $2 \cdot \pi \cdot r \cdot h + \pi \cdot r^2 + \pi \cdot r^2$
 Volume V : $\pi \cdot r^2 \cdot h$
 $V = \pi \cdot r^2 \cdot h$ solve, $h \rightarrow \frac{V}{\pi \cdot r^2}$ $h = \frac{V}{\pi \cdot r^2}$

$S = 2 \cdot \pi \cdot r \cdot h + \pi \cdot r^2 + \pi \cdot r^2$ substitute, $h = \frac{V}{\pi \cdot r^2} \rightarrow S = \frac{2 \cdot (\pi \cdot r^3 + V)}{r}$

$\frac{d}{dr} \frac{2 \cdot (\pi \cdot r^3 + V)}{r} \rightarrow 6 \cdot \pi \cdot r - \frac{2 \cdot \pi \cdot r^3 + 2 \cdot V}{r^2}$

$6 \cdot \pi \cdot r - \frac{2 \cdot \pi \cdot r^3 + 2 \cdot V}{r^2} = 0$ solve, $r \rightarrow$

$$\left[\begin{array}{c} \left(\frac{V}{2 \cdot \pi}\right)^{\frac{1}{3}} \\ \frac{\left(\frac{V}{2 \cdot \pi}\right)^{\frac{1}{3}}}{2} + \frac{\sqrt{3} \cdot \left(\frac{V}{2 \cdot \pi}\right)^{\frac{1}{3}}}{2} \cdot i \\ \frac{\left(\frac{V}{2 \cdot \pi}\right)^{\frac{1}{3}}}{2} - \frac{\sqrt{3} \cdot \left(\frac{V}{2 \cdot \pi}\right)^{\frac{1}{3}}}{2} \cdot i \end{array} \right]$$

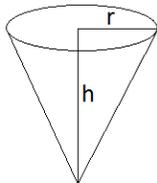
$r_{\text{opt}}(V) = \left(\frac{V}{2 \cdot \pi}\right)^{\frac{1}{3}}$ $h_{\text{opt}}(V) = \frac{V}{\pi \cdot r_{\text{opt}}(V)^2}$ $\frac{h_{\text{opt}}(V)}{r_{\text{opt}}(V)}$ simplify $\rightarrow 2$

Figure 3: The problem of a cylinder with a minimum surface area (Mathcad).

derive an expression for the total surface area S as a function of the variable r and volume V , as well as for its derivative with respect to r .

Well known, the surface area reaches the desired minimum value at a critical point where the derivative is zero or does not exist. The derivative is defined everywhere except at $r = 0$, but a cylinder with zero radius is not sensible. So, we'll find the desired radius if we equate derivative to zero. Mathcad helps us to find the only root of the equation $dS/dr = 0$. Common sense (or a study of where the derivative changes the sign) says that this is a desired value. The ratio h/r equals 2 (what we were going to show) and is independent of volume V .

And what is the optimal ratio of h/r in case of a conical shape container (here r is the radius of the base of the cone)? Such conical silos are used to store the crushed coal for combustion in a furnace of a boiler. This optimization problem is solved here using Maple—another mathematical program—see Figure 4 below.



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> Vol := (r, h) -> 1/3 * pi * r^2 * h
      Vol := (r, h) -> 1/3 * pi * r^2 * h
> Surf := (r, h) -> pi * r * sqrt(r^2 + h^2)
      Surf := (r, h) -> pi * r * sqrt(r^2 + h^2)
> H := unapply(solve(Vol(r, h) = V, h), r, h)
      H := (r, h) -> 3 * V / (pi * r^2)
> S := unapply(subs(h = H(r, V), Surf(r, h)), r, V)
      S := (r, V) -> pi * r * sqrt(r^2 + 9 * V^2 / (pi^2 * r^4))
> Sol := unapply(D[S](r, V), r, V);
      Sol1 := unapply(simplify(Sol(r, V)), r, V) assuming r > 0;
Sol := (r, V) -> pi * sqrt(r^2 + 9 * V^2 / (pi^2 * r^4)) + 1/2 * (pi * r * (2 * r - 36 * V^2 / (pi^2 * r^5)) / sqrt(r^2 + 9 * V^2 / (pi^2 * r^4)))
      Sol1 := (r, V) -> (2 * r^6 * pi^2 - 9 * V^2) / (r^2 * sqrt(r^6 * pi^2 + 9 * V^2))
> R := solve(Sol1(r, V) = 0, r, useassumptions)
      assuming r > 0, V > 0; Ropt := R;
      R := 1/2 * (3^(1/3) * 2^(5/6) * (V^2 * pi^4)^(1/6)) / pi
> Hopt := simplify(subs(r = Ropt, H(r, V))) assuming V > 0;
      Hopt := (V^(1/3) * 3^(1/3) * 2^(1/3)) / (pi^(1/3))
> simplify(Hopt/Ropt) assuming V > 0; sqrt(2)

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Figure 4: The problem of a cone with a minimum surface area (Maple).

And what happens if the top cover of the cylinder is removed and the cylinder is covered with a half sphere? And what happens if the cone has a flat or a spherical cap? Students attacked these and other similar optimization

problems for containers of various shapes with enthusiasm. As a result, they have mastered the basics of mathematical analysis, as well as related tools and Mathcad. They all got a kick out of this work. (These problems and others are available on the site <http://communities.ptc.com/groups/optimisation-with-mathcad>.)

The students notice several features of these problems while solving them using the computer. Why does the formula for the volume of a cone have a coefficient $1/3$ (see the first operator in Figure 2) but not $1/2$ or $1/4$? This fact we explain by imagining a cone, made in the style of children's pyramid of low cylinders with decreasing diameters. The volumes of the low cylinders can be summed in order to obtain an approximate value of the volume of a cone. Then, by reducing the heights of the cylinders, and by increasing their number to infinity (we keep the constant volume) in the limit we get the integral. And the coefficient $1/3$ appears naturally when we calculate the integral. By the way we remember that the sign of integral - a stretched letter s - is the start letter of the Latin word for "sum".

Let us return to reasons. There is one more situation that needs to be discussed.

From time to time one of the authors teaches students to explore the function and build the graph. It is one of the most complex mathematical topics in the first semester. Usually teachers require the students to do that "by hand", because the students must know all the specific features of a process. But much better it will be, if students check by computers the calculation of limits and derivatives. You can also permit to draw a graph by computer program instead of a sketch drawn in by hand.

Work of this nature gives undeniable positive effects. Student proceeds to complex and unusual for him job psychologically relaxed. He does not "deceive the teacher", when he is trying to look for computer tips. He has not fear that the job will not be done in time, or the answer will be wrong. This allows him to focus on the essence of the problem, but not on the estimation for the work. Students accustomed to the fact that complex multistep calculations can and should be checked at every convenient stage. Because it leads to a smaller loss of time to solve the problem correctly. And finally, what is particularly valuable for the education of future engineers. With discrepancies in the results of its own research and computer checks they should decide who is wrong: human or computer.

As a result of such a comprehensive study of the math problem students not only get acquainted with various mathematical tools. They also realize that all of these computer programs are done according to specific scenarios based on numerical methods. The programs have certain restrictions on the parameters and the region of use.

We have analyzed the use of mathematical computer programs by students of our university and concluded that in the interaction of students with the computers anarchy reigns.

Unsystematic, uncontrolled and accidental use of the math programs does not lead to improvement of mathematical preparation or growth of achievers [5]. In order to gain progress in learning math with the assistance of math programs the students must use them under the direct guidance of the teacher [3, 4].

The teachers on mathematics should allow students to use symbolic calculators and computers, shifting the emphasis from the artificial exercises to a deeper understanding of math tools and their practical application to engineering.

Much of the time, students still learn mathematics using the style of textbooks and books of problems of the XVIII century. Yes, there are special courses in higher mathematics with the use of modern computer math programs [14, 18]. But the majority of pupils and students continue to learn according to curriculum and problems that were three hundred years ago.

With the development of computational tools of increased power and with availability of computers and new programs, numerical methods for solving problems should be treated on a par with analytical methods. Scientists usually do so when they solve extremely difficult problems. Now the students can do so when they solve ordinary math problems.

Figure 5 shows the solution of the problem of minimal surface of the cone with hemispherical covering using a combination of analytical transformations (symbolic math) and numerical methods.

After we did calculations for the cylinder (Figure 3) and a single cone (Figure 4) we can continue to investigate another shapes of tanks. We may take a cone covered with a hemispherical top (Figure 5). Of course, it is very interesting for students to derive a formula for the optimal ratio h/r for a cone with a hemisphere. But symbolic mathematics does not always work

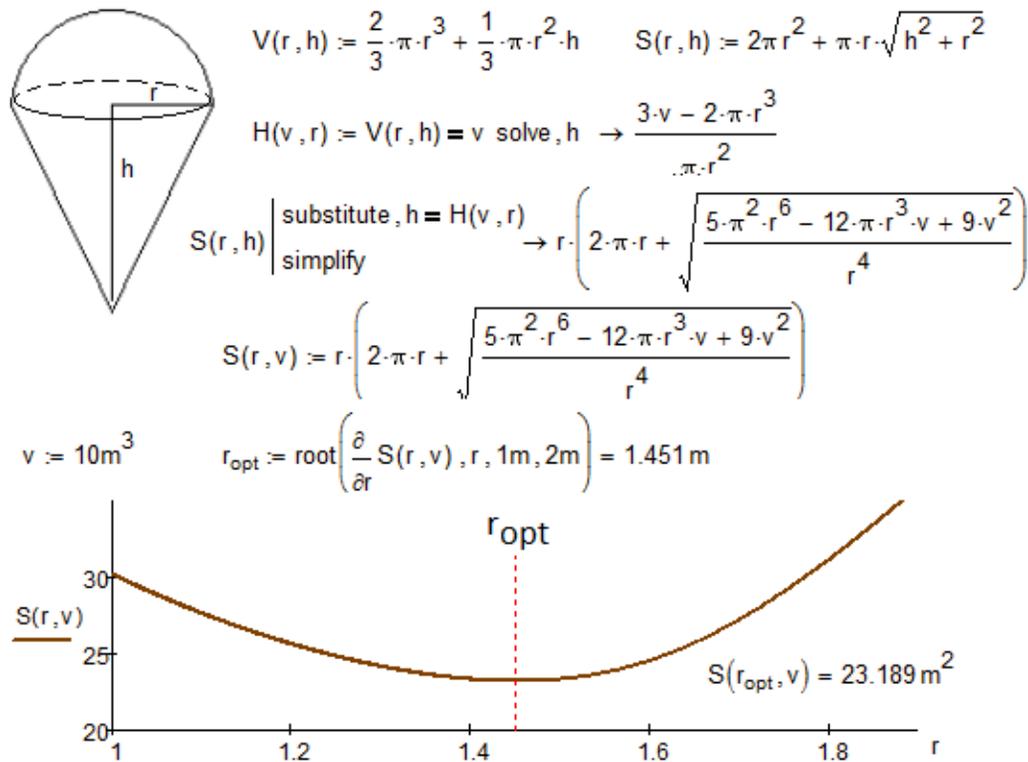


Figure 5: The problem of a cone and a hemisphere with a minimum surface area (Mathcad).

in case of more complicated problems. It also can give very cumbersome solutions. In Figure 5 the problem of the cone covered with a hemispherical top, is solved numerically for the volume of 10 m³. The answer is verified on a graph of the function, which was derived using the symbolic mathematics of Mathcad. A discussion of this problem and its analytical solution using various mathematical software packages (Mathcad, Maple, Mathematica and Derive) is available at the forum <http://communities.ptc.com/message/197522>.

The question about the use of real tasks or abstract math examples remains debatable as a hundred years ago [1, 15]. Primary school pupils learn the basics of mathematics (arithmetic) using real examples. For instance: “You have two apples in your pocket. Someone took one of them from you. How many apples are left?”. Later, in high school and in college real-life

examples are gradually replaced by abstract notions. You can't say anything about the nature of solutions of equations or systems of equations. The "physics" of problems is completely ignored now.

During the last 30–40 years, there was a widespread transition from analytical solutions to numerical approximate methods in engineering. This was due to the limited applicability of analytical methods and to development of computer technology. Thirty years ago any particular problem had to be solved only in a special computer laboratory. Today this can be done with any smartphone. There are new and more complex tasks that smartphone cannot solve. Such problems can be sent via the Internet for solutions on the new supercomputer of the same data center.

But, we repeat, the teaching of mathematics at the Engineering University is still based on the math books in the style of three hundred years ago. Teachers of mathematics must not ignore computer math tools.

4. Teaching Mathematics versus Computer Science

One more reason for this article: The students studied two parallel courses: mathematics and computer science. The course Computer science was based on the use of mathematical software (see <http://twt.mpei.ac.ru/ochkov/Potoki.htm>). At a consultation before an exam in computer science students admitted that they got bad results on the examination on linear algebra. The matter is, that one of the aims of Linear Algebra is to solve systems of linear algebraic equations. Analyzing students' failures on the exam, we used Mathcad and explored a system of three linear equations. They were provided with geometric interpretations (see Figures 6, 7, 8).

Note. The expression $\text{rank}(A) = \text{rank}(A1) = 1$ does not mean that the ranks of A and $A1$ are equal to one, that means the equality of ranks. It is a Boolean expression for true fact.

Figures 6,7, and 8 (Mathcad) show three cases that can appear when we solve systems of three (or more) of linear algebraic equations. An infinite number of solutions (three planes intersect along a straight line - Figure 6). No solutions (three planes intersect only in pairs - Figure 7). A unique solution (three planes intersect in one point - Figure 8). After such analysis of the problem with its graphical interpretation the students told, that if they saw these fragments earlier, then results would be better. The computer

A system of linear equations with n variables has a solution if and only if the rank of its coefficient matrix A is equal to the rank of its augmented matrix $A1$.

$$A \cdot x = B \quad A := \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad B := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad A1 := \text{augment}(A, B) = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{pmatrix}$$

$$\text{rank}(A) = 2$$

$$\text{rank}(A1) = 2$$

$$\text{cols}(A) = 3$$

$$\text{rank}(A) = \text{rank}(A1)$$

$$\text{rank}(A) < \text{cols}(A)$$

$$z1(x, y) := A_{0,0}x + A_{0,1}y + A_{0,2}z = B_0 \quad \text{solve, } z \rightarrow \frac{1}{3} - \frac{2 \cdot y}{3} - \frac{x}{3}$$

$$z2(x, y) := A_{1,0}x + A_{1,1}y + A_{1,2}z = B_1 \quad \text{solve, } z \rightarrow \frac{1}{3} - \frac{5 \cdot y}{6} - \frac{2 \cdot x}{3}$$

$$z3(x, y) := A_{2,0}x + A_{2,1}y + A_{2,2}z = B_2 \quad \text{solve, } z \rightarrow \frac{1}{3} - \frac{8 \cdot y}{9} - \frac{7 \cdot x}{9}$$

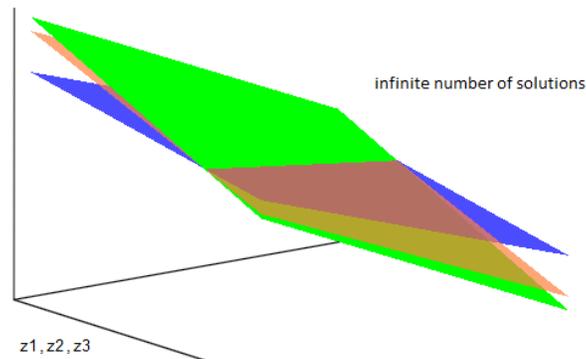


Figure 6: Linear algebraic solution in Mathcad - an infinite number of solutions.

math model helps students to understand and to memorize the theorems for linear algebraic equations.

By the way, the establishment and implementation of a mathematical model for a single power supply of a village, town, or the country as a whole, requires constructing and solving linear algebraic equations with dozens or even hundreds of thousands of unknowns. Large-scale systems are solved by numerical methods of linear algebra, the study of which is not included in a standard course in linear algebra. Such systems can be solved only by means of math computer programs.

The ability to visualize solutions, using static pictures or even animations, is very important. It helps to understand the math theory. Modern math programs provide users with easy and convenient means of animation. For example, the forum Planet PTC contains pages on which are placed the animation solutions of some typical problems of mathematics - see <http://>

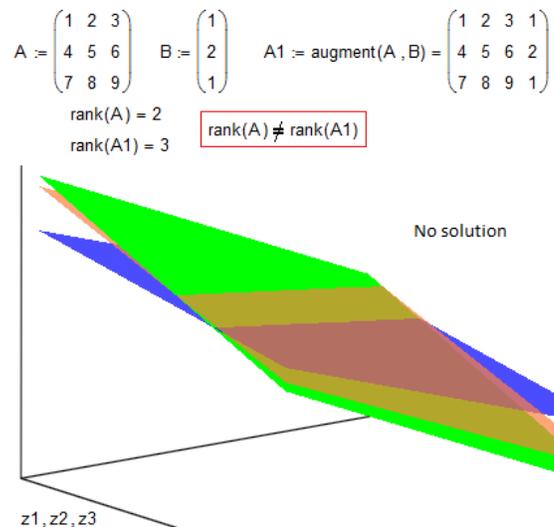


Figure 7: Linear algebraic solution in the medium Mathcad—no solutions.

communities.ptc.com/groups/animation-of-math-methods-in-mathcad. At the same forum <http://communities.ptc.com> there are a lot of three-dimensional models that have been created in CAD package, Creo (previously called PRO/Engineer), and plotted in Mathcad.

The main goal of mathematical education should be nurturing skills mathematically explore real phenomena [2]. Math should not be reduced to recipes. More precisely, the essence of learning math is not a memorizing of theorems and rules. Though they are extremely important. Most of the population never needs a theoretical justification of their actions. People just do.

For that purpose they need to have confidence that everything turns out according to the laws of nature. But how to see the effect of these laws? Mathematics and computer math models help.

For most of the people need not lemmas or formulas. The mathematics must be alive. And formulas can be delivered to computers.

For example, an expert on the packaging of goods in cubistic boxes seeks for optimal way of cutting cardboard. Will he learn a complex mathematical theorem proving that a cube has exactly eleven scans? Most likely, he'll understand this, seeing math video [7]. Then he makes good choose.

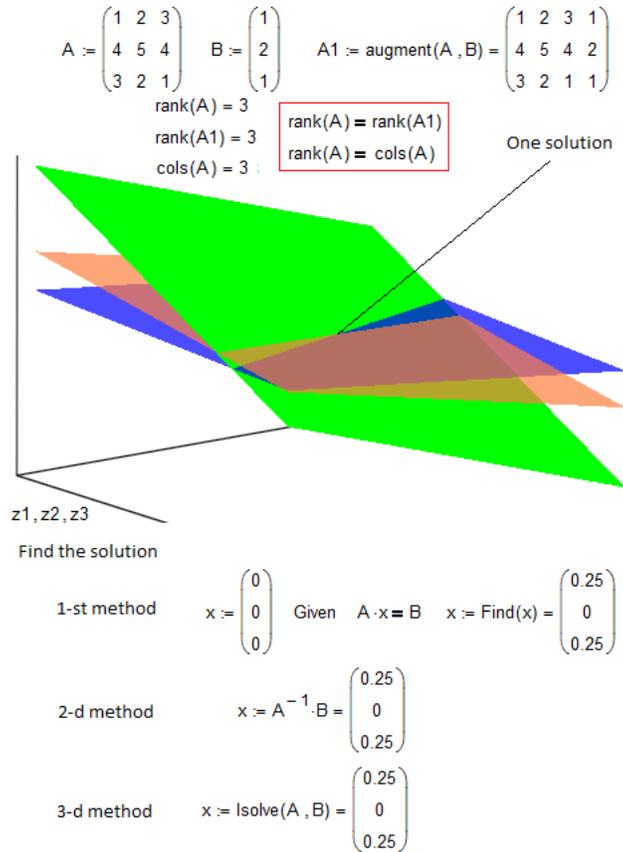


Figure 8: Linear algebraic solution in the medium Mathcad—the only solution.

Lecturer in mathematics needs to deliver many preliminary lectures, before he can prove a well known property of a parabolic antenna. But if future student's profession is not mathematics, then it is enough to show him the math short film [9].

The proof of the theorem named after a French mathematician G. Monge is not easy. But the result of this theorem is often useful in architecture. How can students get acquainted with it? The most effective way is [8]. The same applies to the continuity problems [10].

But we must remember that, despite the existence of a plurality of digital and dynamic representations of mathematical objects, the main problem is the development of mental abilities of our students [17].

5. Conclusions

Conservative colleagues say that the calculator and the computer are the culprits behind poor math knowledge of pupils and students. In particular, it is alleged that calculator and computer can manipulate only decimal fractions, but the arithmetic should use simple fractions. This is a strong reason for future mathematicians, but not for all others. The modern student always gets a calculator at hand, if he can't calculate orally. He does not understand why he must learn to use these simple fractions.

What is there to argue, or, rather, to add?

When the authors were getting their education—secondary and higher education—not more than 30% of pupils went to study in the universities. Now, more than 90% of graduates of secondary schools want to gain the higher education. It isn't bad or good, it's a reality. What kind of mathematical education can we have in such circumstances? We try to analyze the problem.

And what about simple and decimal fractions? Mathcad has tools to work with simple fractions and, if desired, they can be successfully used in elementary school. We get the main effect when, working with simple fractions, we look for the greatest common divisor (GCD) of the denominators. In Mathcad this is a function, called "gcd". Furthermore, in Mathcad one can decompose a number into prime factors (e.g. $69 = 3 \cdot 23$ and $57 = 3 \cdot 19$). Figure ?? shows how you can semi-automatically correctly fold two simple fractions in Mathcad. It also shows the operator panel "Calculator", which allows having a number in the form of Mixed Number. This allows having a very productive way to work with simple fractions in Mathcad. One can use it even in the early grades of school.

Some of our colleagues admit that a computer is useful for lectures on mathematics. But they do not allow the modern mathematical software packages that we speak about in the article.

There is one more, psychological reason to involve the computer to the teaching of math.

It isn't a secret that teachers of mathematics for many pupils and students are not only "excellent teachers", but also "torturers", that force students to memorize complicated theorems and to perform mental arithmetic. Many people still have nightmares about mathematical instruments of torture. To

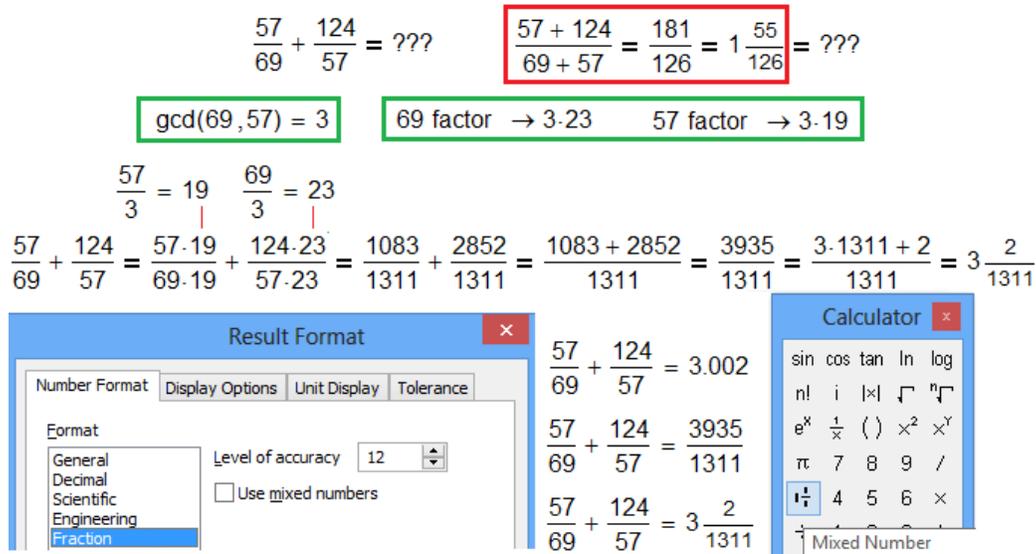


Figure 9: Work in Mathcad with simple fractions.

avoid this computer can take the routine math work (remove the tedious side of mathematics), allowing the teacher and students to create something more fascinating.

1. Advanced mathematical computer programs allow using a fresh approach to the teaching of mathematics in schools and universities, taking into account the attraction of pupils and students to computers.
2. By means of graphics and animation, one can significantly increase the understanding of pupils and students of the basic concepts and theorems of mathematics.
3. Modern information technologies can transform and change the traditional solutions of mathematical problems.
4. In order to make progress in the influence of the computer to the process of learning mathematics the teacher must exercise direct guidance.

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